7.1

A Rational Existence

Introduction to Rational Functions

LEARNING GOALS

In this lesson, you will:

- Graph rational functions.
- Compare rational functions in multiple representations.
- Compare the basic rational function to various basic polynomial functions.
- Analyze the key characteristics of rational functions.

KEY TERMS

- rational function
- vertical asymptote

Consider the following mathematical explanation that 1 is equal to 2. Yes, you read that correctly–you will analyze a proof of 1 = 2.

Let's start by noting that any number multiplied by 0 is equal to 0, correct? Therefore,

$$1 \times 0 = 0$$
$$2 \times 0 = 0.$$

Since the expressions 1×0 and 2×0 both equal zero, then they must be equal to each other by the transitive property. Therefore,

$$1 \times 0 = 2 \times 0.$$

Dividing both sides of an equation by the same value preserves equality. Therefore, you can divide both sides of the equation by 0.

$$\frac{1 \times 0}{0} = \frac{2 \times 0}{0}$$

Anything divided by itself is 1, so $\frac{0}{0} = 1$. This leaves

$$1 \times 1 = 2 \times 1$$
$$1 = 2.$$

There weren't any sneaky tricks or magical sleights of hand in this proof. In fact, all steps were justified according to the rules of algebra . . . so then what's wrong with this proof?

PROBLEM 1 My World Is Turned Upside Down

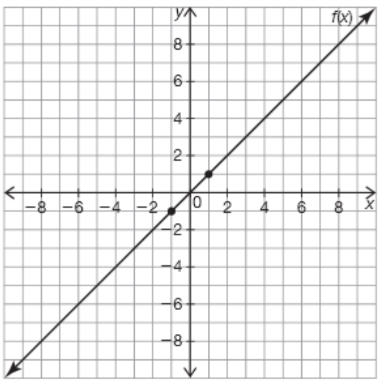
Recall from previous math courses that the reciprocal of any number x is $\frac{1}{x}$.
For example, the reciprocal of 5 is $\frac{1}{5}$ and the reciprocal of 0.5 is $\frac{1}{0.5}$, or 2.

Throughout this course you have studied many connections between polynomial functions and real numbers. Does it follow then that polynomial functions also have reciprocals? Is the reciprocal also a polynomial? Is it a function? How would the graph and table of values of $\frac{1}{f(x)}$ compare to the original function $f(x)$?

To begin answering these questions, consider the reciprocal of the basic linear function $f(x) = x$. The reciprocal can be defined as $g(x) = \frac{1}{f(x)}$, or simply $g(x) = \frac{1}{x}$.

1. Consider the graph and table of values for $f(x) = x$. The domain of $f(x)$ is $(-\infty, \infty)$. The points $(-1, -1)$ and $(1, 1)$ are shown and used to create three intervals for analysis.

x	1	2	3	4	5	6
$f(x) = x$	1	2	3	4	5	6
$g(x) = \frac{1}{x}$						



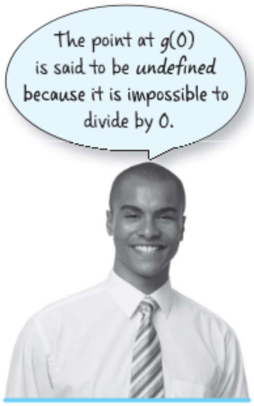
−6	−5	−4	−3	−2	−1	x
−6	−5	−4	−3	−2	−1	$f(x) = x$
						$g(x) = \frac{1}{x}$

x	−1	$-\frac{1}{2}$	$-\frac{1}{10}$	$-\frac{1}{100}$	0	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	1
$f(x) = x$	−1	$-\frac{1}{2}$	$-\frac{1}{10}$	$-\frac{1}{100}$	0	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	1
$g(x) = \frac{1}{x}$									

a. Complete the table of values for $g(x) = \frac{1}{x}$. Then plot the points and draw a smooth curve to graph $g(x)$ on the coordinate plane.

b. Describe the graph of $g(x)$. How is it similar to the graphs of other functions that you've studied? How is it different?

c. Describe the end behavior of $g(x)$. Explain your reasoning in terms of the graph, equation, and table of values.



d. Describe $g(x)$ as x approaches 0 from the left. Explain the output behavior of the function in terms of the equation.

e. Describe $g(x)$ as x approaches 0 from the right. Explain the output behavior of the function in terms of the equation.

2. Henry and Rosie disagree about $g(x) = \frac{1}{x}$.

Henry
The graph and table both clearly show that it is a function.

Rosie
It is not a function. Every input doesn't have an output.

Who is correct? Explain your reasoning.

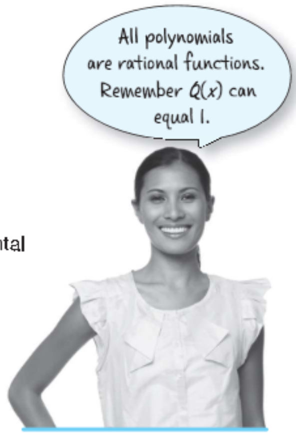
3. Analyze the key characteristics of $g(x) = \frac{1}{x}$.

a. Will the graph ever intersect the horizontal line $y = 0$? Explain your reasoning in terms of the graph, table, and equation.

b. Will the graph ever intersect the vertical line $x = 0$? Explain your reasoning in terms of the graph, table, and equation.

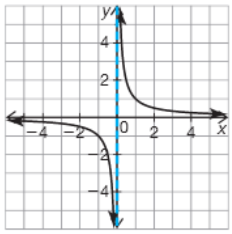
c. Describe the domain and range of $g(x)$.

The function $g(x) = \frac{1}{x}$ is an example of a *rational function*. A **rational function** is any function that can be written as the ratio of two polynomials. It can be written in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x) \neq 0$. You have already seen some specific types of rational functions. Linear, quadratic, cubic, and higher order polynomial functions are types of rational functions.

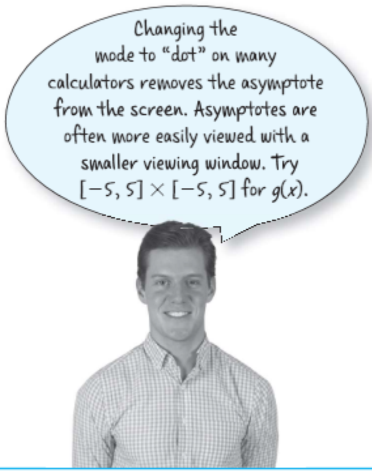


Recall from your study of exponential functions that a horizontal asymptote is a horizontal line that a function gets closer and closer to, but never intersects. In this problem, the function $g(x)$ has a horizontal asymptote at $y = 0$.

The function $g(x) = \frac{1}{x}$ has a *vertical asymptote* at $x = 0$. A **vertical asymptote** is a vertical line that a function gets closer and closer to, but never intersects. The asymptote does not represent points on the graph of the function. It represents the output value that the graph approaches. An asymptote occurs for input values that result in a denominator of 0.



The vertical asymptote is often represented in textbooks and graphing calculators as a dashed or solid line. The convention used in this textbook is to represent asymptotes as dashed lines.



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4. Analyze each function.

$f(x) = x$	$g(x) = \frac{3x}{2}$	$h(x) = \frac{\sqrt{x}}{2x}$
$p(x) = \frac{3}{x} + 2$	$k(x) = 12$	$n(x) = \frac{2^x}{5}$
$j(x) = \frac{4x^2 + 3x + 2}{6x^3 + 10}$	$m(x) = \frac{1}{(x + 2)(x - 3)}$	

a. Circle the rational functions.

b. Explain why the remaining functions are not rational.

c. Do you think the graphs of all rational functions will have a vertical asymptote? Explain your reasoning.

5. Use a graphing calculator to explore various rational functions of the form $p(x) = \frac{a}{x}$ where a is a constant.

a. Describe changes in the function for various a-values. Make as many conjectures as you can about the key characteristics of functions in this form.

b. Abby and Natasha disagree about functions of the form $p(x) = \frac{a}{x}$ where a is a constant.

Abby	Natasha
The horizontal asymptote will vary depending on the a-value.	All rational functions of this form will have a horizontal asymptote at $y = 0$.

Who is correct? Explain your reasoning.

c. List several rational functions that do not have a horizontal asymptote at $y = 0$.

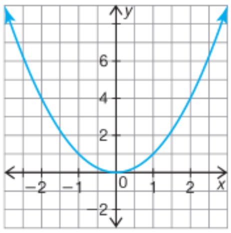
6. If $g(x)$ is the reciprocal function of $f(x)$, what is $f(x) \cdot g(x)$ where $g(x) \neq 0$ for any input value? Explain your reasoning.

7. The opener to this lesson provides a proof that $1 = 2$. Describe the error in the proof in terms of what you learned in this problem.

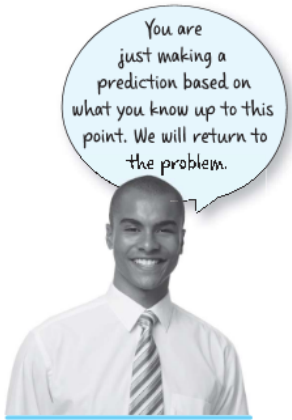
PROBLEM 2 Power in Rational Behavior

Recall that power functions are any functions of the form $y = x^n$ for $n \geq 1$. In Problem 1, *My World Is Turned Upside Down*, you discovered that the graph of the function $g(x) = \frac{1}{x}$ looks very different than the linear function $f(x) = x$. How will the graphs of the other power functions compare to their reciprocals? Will they all have the same shape? Will they all have asymptotes?

1. Analyze the graph of the quadratic power function $q(x) = x^2$.



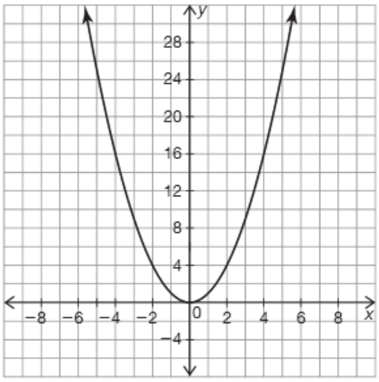
Predict the graph of $r(x) = \frac{1}{x^2}$. Sketch it on the coordinate plane. Explain your reasoning.



2. Consider the graph and table of values for $q(x) = x^2$. The domain of $q(x)$ is $(-\infty, \infty)$. The tables represent three intervals of the domain.

-5	-4	-3	-2	-1	x
		9	4	1	$q(x) = x^2$
					$r(x) = \frac{1}{x^2}$

x	1	2	3	4	5
$q(x) = x^2$	1	4	9		
$r(x) = \frac{1}{x^2}$					



x	-1	$-\frac{1}{2}$	$-\frac{1}{10}$	$-\frac{1}{100}$	0	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	1
$q(x) = x^2$									
$r(x) = \frac{1}{x^2}$									

a. Complete the table of values for $r(x) = \frac{1}{x^2}$.

b. Plot the points and sketch the reciprocal function $r(x)$ on the coordinate plane.

c. Describe the shape of the graph of $r(x) = \frac{1}{x^2}$. How is it similar to $g(x) = \frac{1}{x}$? How is it different?

3. Analyze the key characteristics of $r(x)$.

   a. Describe the domain and range of $r(x)$.

   b. Describe the end behavior of $r(x)$.

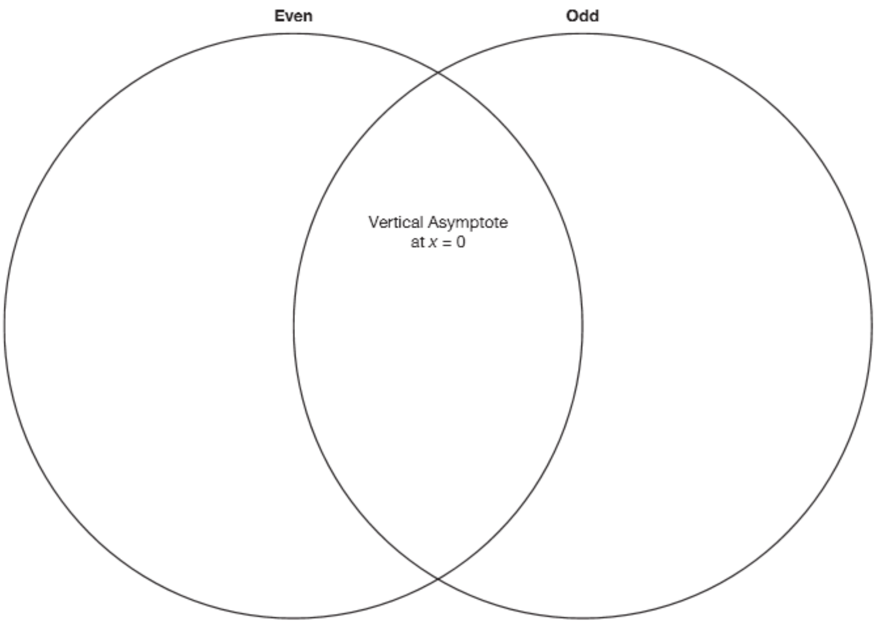
   c. Describe the horizontal and vertical asymptotes of $r(x)$. How can you determine the asymptotes from the graph, table, and equation?

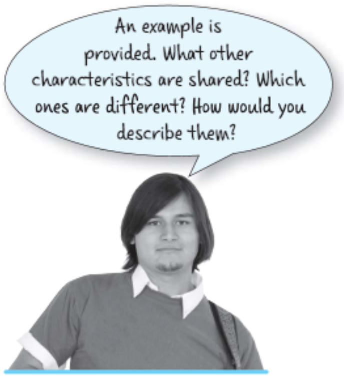
4. Use a graphing calculator to explore the key characteristics of the reciprocals of all power functions. Consider the general shape of the graphs, domain, range, end behavior, horizontal asymptotes, and vertical asymptotes.

   a. List your conjectures about the even-powered functions $\left\{\frac{1}{x^2}, \frac{1}{x^4}, \frac{1}{x^6}, \ldots\right\}$. Justify your conjectures.

   b. List your conjectures about the odd-powered functions $\left\{\frac{1}{x^3}, \frac{1}{x^5}, \frac{1}{x^7}, \ldots\right\}$. Justify your conjectures.

c. Summarize the similarities and differences between the groups of reciprocal power functions by describing the key characteristics in the Venn diagram. Characteristics that are shared should go in the overlapping space.





Be prepared to share your solutions and methods.

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